Lecture 4:

Recall: Analytic Spectral method to solve

$$L u(x) = g(x).$$

Find basis functions $\{\phi_n(x)\}_{n=1}^{\infty}$ such that:

$$L\phi_n(x) = \sum_{j=1}^{N} \lambda_j^n \phi_j(x)$$

$$g(x) \approx \sum_{j=1}^{N} b_j \phi_j(x).$$

Let $u(x) = \sum_{j=1}^{N} a_j \phi_j(x)$.

Then: $L u(x) = g(x) \Rightarrow \sum_{j=1}^{N} a_j \sum_{k=1}^{N} \lambda_k^j \phi_k(x) = \sum_{j=1}^{N} b_j \phi_j(x)$

Comparing coefficients $\Rightarrow$ Diff. eqt becomes algebraic eqts.

<u>Remark:</u> 1. By writing $u(x)$ as linear combination of basis functions (eigenfunctions), complicated differential equation can be converted to algebraic equation.

2. Spectral method is related to eigenvalues and eigenfunctions of some differential operators (e.g. $\sin x$ is eigenfunction of $\frac{d^2}{dx^2}$)

It's called Spectral decomposition of differential operator.

Example: Consider: $u_t = u_{xx}$, $x \in [0, 2\pi]$ such that

$$u(0,t) = u(2\pi, t) \quad \text{(periodic)}$$
$$u(x,0) = f(x) \quad \text{(initial condition)}$$

Solution: Let $u(x,t) = X(x)\,T(t)$

Consider $L = \frac{\partial^2}{\partial x^2}$. Construct $\{\phi_n(x)\}_{n=1}^{\infty} = \{\cos nx, \sin nx, \cancel{e^{nx}}\}_{n=0}^{\infty}$

BUT: $u(0,t) = u(2\pi, t) \Rightarrow X(0)\,T(t) = X(2\pi)\,T(t)$
$\Rightarrow X(0) = X(2\pi)$

X must be periodic. $\therefore$ e^{kx} CANNOT be the choice!!

Let $u(x,t) = \sum_{n=1}^{N} a_n(t) \cos nx + b_n(t) \sin nx$

$$u_t = u_{xx} \Rightarrow \sum_{n=1}^{N} a_n'(t) \cos nx + b_n'(t) \sin nx = \sum_{n=1}^{N} (-n^2) a_n(t) \cos nx + (-n^2) b_n(t) \sin nx$$

Comparing coefficients: $a_n'(t) = -n^2 a_n(t)$ and $b_n'(t) = -n^2 b_n(t)$.

Solving $a_n'(t) = -n^2 a_n(t) \Rightarrow a_n(t) = a_n e^{-n^2 t}$ $(a_n \in \mathbb{R})$

Similarly, $b_n(t) = b_n e^{-n^2 t}$ $(b_n \in \mathbb{R})$

$$\therefore \quad u(x,t) = \sum_{n=1}^{N} a_n e^{-n^2 t} \sin nx + \sum_{n=1}^{N} b_n e^{-n^2 t} \cos nx$$

How to determine a_k and b_k ?? Initial condition: $u(x,0) = f(x)$.

Suppose $f(x) = \sum_{k=0}^{\infty} c_k \cos kx + d_k \sin kx$.

Then: $u(x,0) = f(x)$ implies:

$$\sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx = \sum_{k=0}^{\infty} c_k \cos kx + d_k \sin kx$$

Comparing coefficients:

$$a_k = c_k$$
$$b_k = d_k$$

(Algebraic eqt).

Question: Given $f(x)$, how to find a_k and b_k such that $f(x) = \sum_{k=0}^{\infty} a_k \cos kx + b_k \sin kx$?

(Fourier analysis problem)

Note that: $\int_0^{2\pi} \cos kx \cos mx \, dx = \begin{cases} 2\pi & \text{if } k = m = 0 \\ \pi & \text{if } k = m \neq 0 \\ 0 & \text{if } k \neq m \end{cases}$

e.g. $\int_0^{2\pi} \cos kx \cos kx \, dx = \int_0^{2\pi} \frac{1 + \cos(2kx)}{2} dx = \pi$

Also, $\int_0^{2\pi} \sin kx \sin mx \, dx = \begin{cases} 2\pi & \text{if } k = m = 0 \\ \pi & \text{if } k = m \neq 0 \\ 0 & \text{if } k \neq m \end{cases}$

$\int_0^{2\pi} \sin kx \cos mx \, dx = 0$.

If $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$

For $m > 0$, $\int_0^{2\pi} f(x) \cos mx \, dx = \pi a_m$

$$\therefore \quad a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx$$

$$\int_0^{2\pi} f(x) \sin mx \, dx = \pi b_m$$

$$\therefore \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx \, dx.$$

Also, $\int_0^{2\pi} f(x) \, dx = a_0 (2\pi) \Rightarrow a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx.$

$\therefore$ All a_k, b_k can be computed!!